Dissipative properties and scaling law for a layer of granular material on a vibrating plate

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The dissipative properties and scaling law for a layer of vertically vibrated granular materials were investigated by means of a dynamical model of a single sphere colliding completely inelastically with a massive, oscillating plate. A relationship is presented of how the temperature of the layer scales with the acceleration of the plate and the restitution coefficient of the grains. The numerical calculation shows the existence of an ''energy well'' and a ''temperature well,'' which could be used to explain the existence of a $f/2$ flat (where f is the external driving frequency) state in an experiment on vibrated granular material.

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Experiments have shown that the motion of a granular system consisting of a large number of gains decays rapidly once the energy supply is stopped [1], i.e., the granular system is strongly dissipative, although the grains in the system are only weakly dissipative. Recent research illustrated that the dissipative nature of granular material can result in a variety of different collective phenomena, including heap formation, size segregation, and surface waves [2], which are related to the correlation induced by multiple collisions between the grains and by the relative motion of the granular layer and the container [3]. The dissipation and the scaling law of one- or two-dimensional vibrated granular systems were investigated in experiments [4], as well as kinetic theory [5,6], molecular dynamics simulations [7], a single-particle statistical model [8], etc. Also mentioned when describing the motion of a layer of vibrated granular materials was a model of a single sphere colliding completely inelastically with a massive, sinusoidally oscillating plate [1,3]. Here we use this model to discuss the energy input, the dissipation, and the scaling law for vertically vibrated granular materials. When a granular layer collides with a plate the momentum, as a whole, is conserved, but the kinetic energy of the granular layer is dissipated and transferred into ''internal energy'' of the layer, including the kinetic energy of the grains and the real internal energy due to inelastic collision between grains. In the stationary state the energy input from external excitation is balanced by the dissipation led by the inelastic collisions between grains. The inelastic collision between grains leads to a rapid ''cooling'' and ''clumping'' of the grains with the initial velocities. The energy transferred during the collision is dependent upon the acceleration of the plate.

Suppose that the plate vibrated vertically in the form $A\sin 2\pi ft$ (where A is the driving amplitude and f the driving frequency). We use dimensionless acceleration amplitude $\Gamma = 4\pi^2 f^2 A/g$ (where g is the gravitational acceleration) and a driving frequency f as control parameters. When $\Gamma<1$ the sphere vibrates together with the plate. As Γ increased from 1, the first, second, third, ... nth ... critical acceleration, $\Gamma_{cn}=[(n\pi)^2+1]^{1/2}$ (where $n=1,2,\ldots$), are reached in sequence, at which point the collision becomes instantaneous, and period doubling occurs for the motion of the sphere. Figure 1 shows the rebound velocity of the sphere as a function of the acceleration of the plate, Γ.

Case I: $1<\Gamma<\Gamma_{c1}$. The trajectories of the sphere and the plate are shown in Fig. 2(a). The period of motion of the sphere, T_p, is the same as that of the plate, $T(=1/f)$. The sphere leaves the plate with a ballistic motion for part of each cycle. In the laboratory reference frame, we denote the precollisional velocity of the sphere by V_c^-, and the plate velocity by u_c. The moment t_s at which the sphere leaves the plate is determined by

$$\Gamma \sin 2\pi f t_s = 1. \tag{1}$$

At this moment the velocity of the sphere, V_s, is

$$V_s = 2\pi f A \cos 2\pi f t_s. \tag{2}$$

In each cycle, the moment t_{c1} at which the collision occur is determined by

$$A\sin 2\pi f t_s + V_s(t_{c1}-t_s) - \tfrac{1}{2}g(t_{c1}-t_s)^2 = A\sin 2\pi f t_{c1}. \tag{3}$$

Then the precollisional velocity of the sphere is

$$V_{c1}^- = V_s - g(t_{c1}-t_s), \tag{4}$$

and at this moment the velocity of the plate is

$$u_{c1} = 2\pi f A \cos 2\pi f t_{c1}. \tag{5}$$

Thus the kinetic energy loss of the sphere due to the complete inelastic collision is

$$\Delta E_1 = \frac{Nm}{2}(3u_{c1}^2 - 4u_{c1}V_{c1}^- + V_{c1}^{-2}), \tag{6}$$

where N is the total number of the grains in the sphere, and m is the mass of each grain. This energy is transferred to the sphere as its ''internal energy.'' Once the sphere receives this energy, it ''warms up'' first, i.e., the averaged velocity of the grains raises to a certain value v_0, and then ''cools down.'' We suppose that the sphere warms up uniformly, and then cools down in the form [9]

$$v(t) = \frac{v_0\lambda}{(1-e^2)\gamma v_0 t + \lambda}, \tag{7}$$

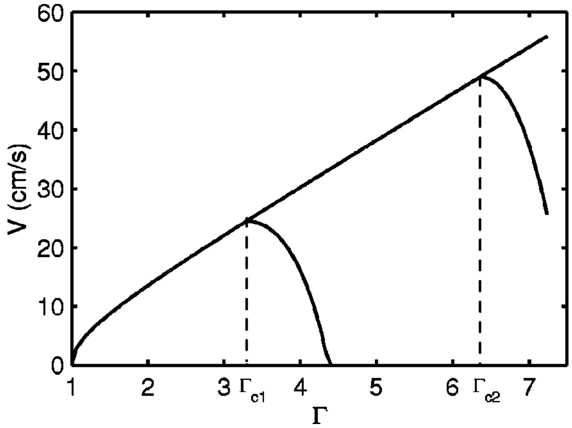

FIG. 1. The rebouncing velocity of a completely inelastic sphere as a function of the acceleration of the vibrating surface.

where e is the restitution coefficient of the grain in the sphere, and λ the mean grain separation in the sphere. Then we must have

$$\frac{Nm}{2}[v_0^2-v^2(T)]=\Delta E_1. \tag{8}$$

From this equation we can obtain the solution to v_0. By averaging $v^2(t)$ in one period T, we can obtain the averaged temperature of the sphere:

$$E_0=\frac{m}{2}\langle v^2\rangle=\frac{m}{2}\,\frac{v_0^2\lambda}{(1-e^2)v_0T+\lambda}. \tag{9}$$

Case II: $\Gamma_{c1}<\Gamma<4.6$. The trajectories of the sphere and the plate are shown in Fig. 2(b). Here $T_p=2T$. In each cycle there are two collisions between the sphere and the plate; one is instantaneous (we denote this as the first collision). For the first collision, the motion of the sphere and the plate are still described by Eqs. (1)–(6). Once the collision occurs the sphere leaves the plate immediately, and performs a ballistic motion. The moment t_{c2} at which the second collision occurs is determined by

$$A\sin 2\pi f t_{c1}+V_{c1}^{+}t_{c21}-\tfrac{1}{2}gt_{c21}^2=A\sin 2\pi f t_{c2}, \tag{10}$$

where $t_{c21}=t_{c2}-t_{c1}$, and $V_{c1}^{+}=u_{c1}$ is the first post-collisional velocity of the sphere. Then the second precollision velocity of the sphere is

$$V_{c2}^{-}=V_{c1}^{+}-gt_{c21}. \tag{11}$$

At this moment, the velocity of the plate is

$$u_{c2}=2\pi fA\cos 2\pi f t_{c2}. \tag{12}$$

Thus the kinetic energy loss of the sphere due to the complete inelastical collision between the sphere and the plate is

$$\Delta E_2=\frac{Nm}{2}(3u_{c2}^2-4u_{c2}V_{c2}^{-}+V_{c2}^{-2}). \tag{13}$$

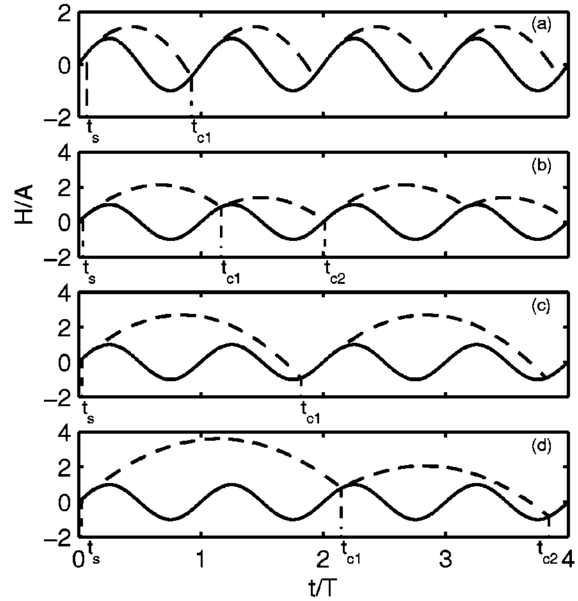

FIG. 2. The trajectories of the sphere (dashed line) and the plate (solid line).

Therefore, in one cycle, the total energy transferred is $\Delta E_1+\Delta E_2$. Similar to case *I*, the averaged temperature of the sphere is

$$E_0=\frac{m}{4T}\left[\frac{v_0^2\lambda t_{c21}}{(1-e^2)v_0t_{c21}+\lambda}+\frac{v_0^2\lambda(2T-t_{c21})}{v_0(1-e^2)(2T-t_{c21})+\lambda}\right]. \tag{14}$$

As Γ is increased from Γ_{c1}, the interval between first and second collisions, t_{c21}, becomes smaller and smaller, and at $\Gamma=4.6$, $t_{c21}=0$, i.e., the second collision disappears. Therefore we have the following

Case III: $4.6<\Gamma<\Gamma_{c2}$. The trajectories of the sphere and the plate are shown in Fig. 2(c). Here $T_p=2T$. In each cycle, there is only one collision between the sphere and the plate. During the collision the kinetic energy loss of the sphere is expressed by ΔE_1. Meanwhile the averaged temperature is

$$E_0=\frac{m}{2}\,\frac{v_0^2\lambda}{2(1-e^2)v_0T+\lambda}. \tag{15}$$

When $\Gamma>\Gamma_{c2}$, the case becomes more complicated than above. Here we consider only the following

Case IV: $\Gamma_{c2}<\Gamma<\Gamma_{cert}$ (where $\Gamma_{c2}<\Gamma_{cert}<\Gamma_{c3}$), in which $T_p=4T$. The trajectories of the sphere and the plate are shown in Fig. 2(d). In each cycle there are two collisions between the sphere and the plate, with one being instantaneous. Similar to case II, the averaged temperature of the sphere is

$$E_0=\frac{m}{8T}\left[\frac{v_0^2\lambda t_{c21}}{(1-e^2)v_0t_{c12}+\lambda}+\frac{v_0^2\lambda(4T-t_{c21})}{v_0(1-e^2)(4T-t_{c21})+\lambda}\right]. \tag{16}$$

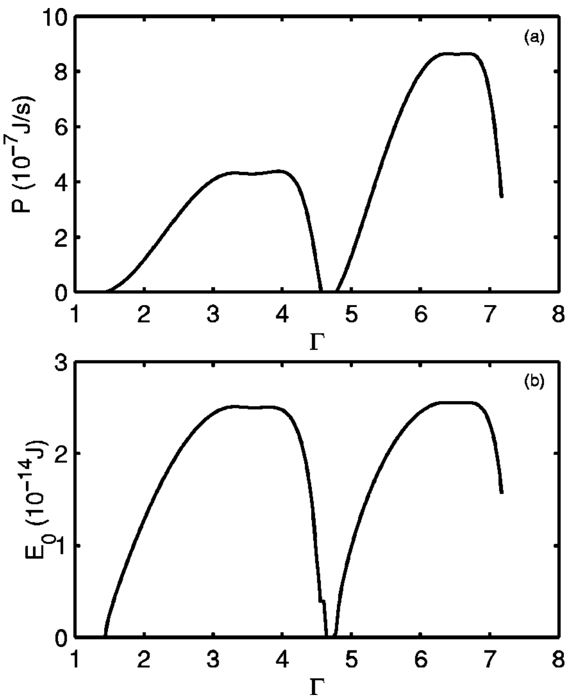

FIG. 3. (a) The mean power input P for each grain, and (b) the temperature of the sphere, E_0, as functions of dimensionless acceleration of the plate.

Equations (9), (14), (15), and (16) show explicitly how E_0 scales with λ and e. It is found that the larger the restitution coefficient e and/or the mean grain separation λ, the higher the temperature of the sphere. These equations also implicitly give the dependence of the temperature of the sphere on the acceleration of the plate, Γ. To see this we have made numerical calculations for all the above equations. We take $e=0.95$, $\lambda=0.001$ mm, $f=20$ Hz, and $m=9\times10^{-5}$ g. The mean power input P (the ratio of the total energy input and the period in each cycle for each grain) and the temperature of the sphere as functions of acceleration of the plate are shown in Fig. 3. We found out that both the power input and the temperature rise as the acceleration of the plate increases from 1. But in the range around $\Gamma=4.6$, both of them fall rapidly to zero; we call them the ''energy well'' and the ''temperature well'' respectively. The existence of the energy well can be explained as follows: as Γ reaches 4.6, the relative velocities of the sphere and the plate become smaller and smaller. When $\Gamma=4.6$, velocities of both the sphere and the plate are -35.035 cm/s, and the relative velocity is zero. Therefore, both the energy input and the temperature of the sphere are zero. The existence of the energy well can be used to explain the existence of the $f/2$ flat state when Γ is in the range of 4.5–5.5 in the experiment, and the numerical simulation on a vertically oscillated grain layer [10]: as the energy input decreases to zero, the amplitude of the pattern decreases to zero, and shows a flat state. In both experiments and the molecular dynamics simulation the central acceleration of the plate for a $f/2$ flat state is little more than 4.6 of our model. We consider this because the granular layer is not a real single complete inelastic sphere, and not all grains collide with and separate from the plate simultaneously; the collision time of the granular layer is larger than that of the single sphere. The center of mass of the granular layer needs to be raised more highly than that of the single sphere in order that all grains can leave the plate for free flight. Thus all characterized accelerations for granular layer have to be higher than all corresponding characterized accelerations for a single sphere. Also, we note that in Fig. 3(a) the second peak of P is higher than the first one as Γ rises but in Fig. 3(b) the heights of both peaks are almost the same. This result can be explained by the fact that the sphere cools in the form of Eq. (7), the larger v_0 is, the larger the rate of the cooling; also, the longer the time of cooling, the lower the averaged temperature of the sphere. Figure 3(b) shows how E_0 scales with Γ, which is different from the results obtained by Warr *et al.* [4] and Luding *et al.* [7]. Although the assumptions of uniform warming and the form of the cooling [Eq. (7)] are approximate, this assumption does not essentially affect the shape of the curves in Fig. 3. Experiments are currently in progress to examine the scaling law presented in this paper.

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[1] Francisco Melo, Paul Umbanhowar, and Harry L. Swinney, Phys. Rev. Lett. **72**, 172 (1994).
[2] H.M. Jaeger, S.R. Nagel, and R.P. Behringer, Phys. Today **49(4)**, 32 (1996).
[3] Francisco Melo, Paul Umbanhowar, and Harry L. Swinney, Phys. Rev. Lett. **75**, 3838 (1994).
[4] Stephen Warr, Jonathan M. Huntley, and George T.H. Jacques, Phys. Rev. E **52**, 5583 (1995).
[5] V. Kumaran, J. Fluid Mech. **364**, 163 (1998).
[6] J.T. Jenkns and S.B. Savage, J. Fluid Mech. **130**, 187 (1983).
[7] S. Luding, H.J. Herrmann, and A. Blumen, Phys. Rev. E **50**, 3100 (1994).
[8] Stephen Warr, and Jonathan M. Huntley, Phys. Rev. E **52**, 5596 (1995).
[9] P.K. Haff, J. Fluid Mech. **134**, 401 (1983).
[10] C. Bizon, M.D. Shattuck, J.B. Swift, W.D. McCormick, and Harry L. Swinney, Phys. Rev. Lett. **80**, 57 (1998).